Graphing Quadratics

Baldwin

Consider the quadratic $y = ax^2 + \underline{bx} + c$.

For now we'll examine quadratics in the form $y = f(x)$ and not $x = f(y)$. The items to look for when graphing a quadratic are

1. y-intercept
2. x-intercepts [zeros]
3. Vertex
4. Focus
5. Directrix
6. Smooth parabolic curve

1] The y-intercept occurs when $x = 0$, and if you plug 0 in for x you will get $y = c$. so the y-intercept is at $(0, c)$.

2] The x-intercepts, or zeros, are at $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, if they exist. $b^2 - 4ac$ is called the discriminant, and if it is positive, then the radical can be taken and there will be two real zeros. If the discriminant is 0, then there will only be 1 real zero, and if it is negative, there will be no real zeros. Here is a proof of why the x-intercepts are at $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$:

The zeros of a quadratic are when $y = 0$, so $0 = ax^2 + bx + c.$. So to complete the square [instructions at the end of this page], I like to factor out the leading term a first. Dividing both sides by a gives us $0 = x^2 + \dfrac{b}{a}x + \dfrac{c}{a}.$ To complete the square I usually take the second term's coefficient, cut it in half, square it, then add and subtract that from the quadratic giving us $0 = x^2 + \dfrac{b}{a}x + \dfrac{b^2}{4a^2} - \dfrac{b^2}{4a^2} + \dfrac{c}{a}.$ That gives us five terms, the first three are a perfect square trinomial, and the last two are combinable fractions.

$0 = \left(x + \dfrac{b}{2a}\right)^2 + \dfrac{-b^2 + 4ac}{4a^2}$. Subtracting from both sides gives us $\left(x + \dfrac{b}{2a}\right)^2 = \dfrac{b^2 - 4ac}{4a^2}$. Take the square roots of both sides and you get $x + \dfrac{b}{2a} = \pm\sqrt{\dfrac{b^2 - 4ac}{4a^2}}$, or $x + \dfrac{b}{2a} = \pm\dfrac{\sqrt{b^2 - 4ac}}{2a}$, and subtracting from both sides gives us $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$, simplifying to

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

3] The vertex is at $V\left(\frac{-b}{2a}, c - \frac{b^2}{4a}\right)$.

The proof for that is that the vertex has an *x*-coordinate that is the midpoint of the *x*-coordinates of the zeros, since a quadratic's parabola is symmetric about the axis of symmetry. So the *x*-coordinate of the vertex must be

$\dfrac{\frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}}{2} = \frac{-2b}{4a} = \frac{-b}{2a}$. The y-coordinate would be achieved by plugging this *x*-value into the function $y = ax^2 + bx + c$, giving us

$$a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c = \left(\frac{ab^2}{4a^2}\right) + \frac{-b^2}{2a} + c = \frac{ab^2}{4a^2} - \frac{2ab^2}{4a^2} + c = c - \frac{b^2}{4a}.$$

4] The focus is derived from the actual definition of a parabola, which is the locus of points equidistant from a line and a point not on the line. The relationship between the leading coefficient *a* and the distance between the vertex and focus is $f = \frac{1}{4a}$.

5] The directrix is that same distance only in the opposite direction from the vertex.

6] Using the points found in the above process, the parabola that describes the quadratic is a smooth curve that satisfies these points. It's not a "U", and it's not a "V", but between the two, a "Vue".

Let's take on an example: $f(x) = 2x^2 - 5x - 12$.

1] The *y*-intercept is (0, -12).

2] The *x*-intercepts are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5 \pm \sqrt{25+96}}{4} = \frac{5 \pm \sqrt{121}}{4} = \frac{5 \pm 11}{4} = \frac{16}{4} \text{ or } \frac{-6}{4} = 4 \text{ or } -\frac{3}{2}$$

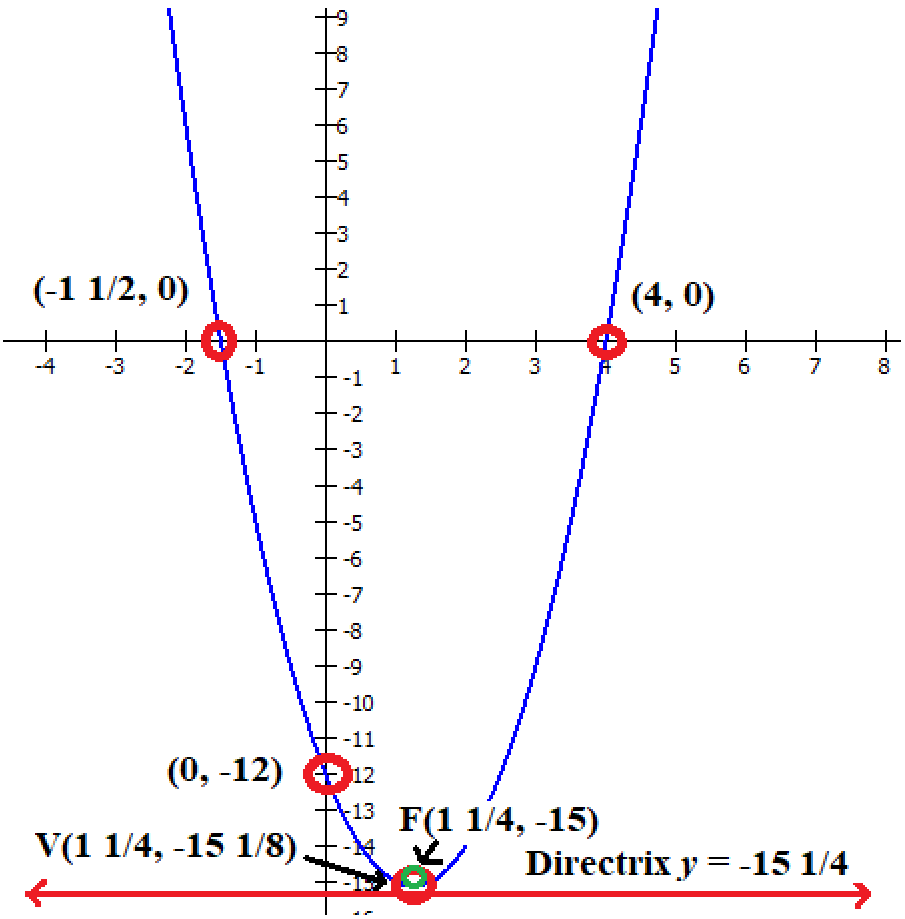
So the *x*-intercepts are $(4, 0)$ and $\left(-\frac{3}{2}, 0\right)$.

3] The vertex is at $V\left(\frac{-b}{2a}, c - \frac{b^2}{4a}\right) = V\left(\frac{5}{4}, -12 - \frac{25}{8}\right) = V\left(1\frac{1}{4}, -15\frac{1}{8}\right)$.

4] The focus is $f = \dfrac{1}{4a}$ from the vertex inside the parabola on the axis of symmetry, or in this case, $V\left(1\frac{1}{4}, -15\frac{1}{8}\right) + \left(0, \frac{1}{8}\right) = F\left(1\frac{1}{4}, -15\right)$.

5] The directrix, which hasn't much of a purpose for what we are doing until we are in conic sections, is a line a distance of *f* on the other side of the vertex than the focus, or $y = -15\frac{1}{8} - \frac{1}{8} = y = -15\frac{1}{4}$.

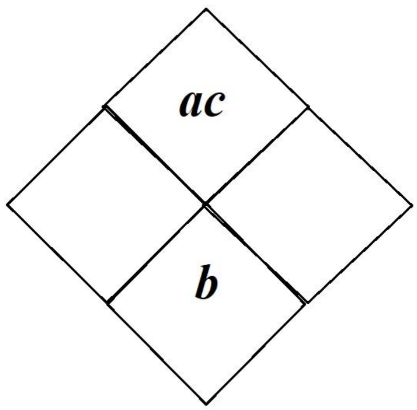
6] Now we graph all this information onto a nice, smooth parabolic representation of the quadratic.

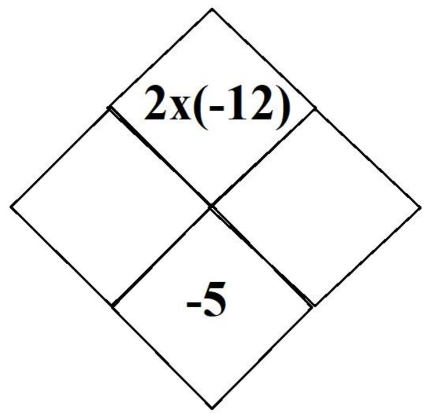


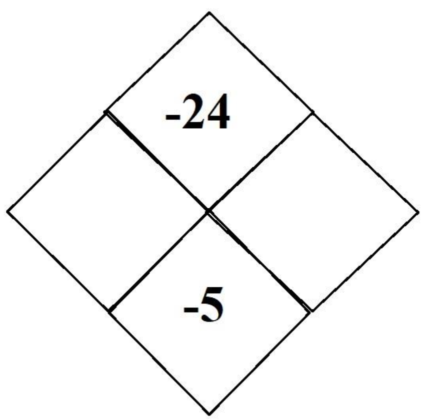
The *y*-intercept and *x*-intercept are in red circles. The vertex is in a red circle and the focus is in a green circle. The directrix is a red line. The parabola itself representing the quadratic function is the blue curve.

A second method for finding the zeros is the **M&M Method**.

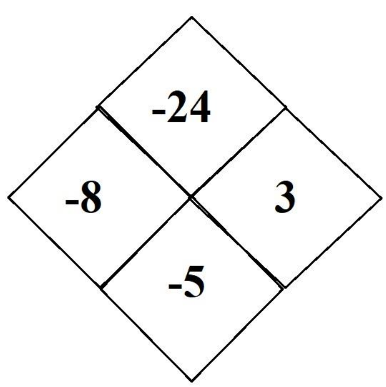
Let's use our quadratic that we used in the first part of this, $y = ax^2 + bx + c$ will be $f(x) = 2x^2 - 5x - 12$. We make a diamond and fill it with the following information:







We need two numbers, *M* and *N*, such that these two numbers multiply to *ac* and add to *b*. The numbers that multiply to -24 are -1&24, -2&12, -3&8, -4&6 -6&4, -8&3, -12&2 and finally, -24&1. If you add these numbers up, only one set adds to -5, and that would be -8 & 3. So -8 and 3 go in the side boxes. Here it is:



We now replace the middle term of the quadratic with these two numbers, so instead of having $f(x) = 2x^2 - 5x - 12$, we now have $2x^2 - 8x + 3x - 12$. The -8 and 3 can go in any order. This can be split down the middle and each side can be factored, giving us $\left(2x^2 - 8x\right) + \left(3x - 12\right) = 2x(x-4) + 3(x-4)$. Recombining, we have $(2x + 3)(x - 4)$. So for x-intercepts, we set the quadratic equal to zero, giving us $0 = (2x + 3)(x - 4)$, and to factor completely, we factor out ALL leading coefficients of the binomial factors, giving us $0 = 2(x + \frac{3}{2})(x - 4)$, and for this to be true, it would force x into being either $-\frac{3}{2}$ or 4.

Completing the Square

0] Before starting the completing the square section, please notice what happens when you square a binomial:

$(x+3)^2 = x^2 + 6x + 9$		$x^2 + 6x + 9 = (x+3)^2$
$(x-2)^2 = x^2 - 4x + 4$		$x^2 - 4x + 4 = (x-2)^2$
$(x+8)^2 = x^2 + 16x + 64$	which leads us to	$x^2 + 16x + 64 = (x+8)^2$
$(x-7)^2 = x^2 - 14x + 49$		$x^2 - 14x + 49 = (x-7)^2$
$(x+12)^2 = x^2 + 24x + 144$		$x^2 + 24x + 144 = (x+12)^2$

Notice the following:

The square root of 9 is 3 and half of 6 is 3.
The square root of 4 is 2 and half of 4 is 2.
The square root of 64 is 8 and half of 16 is 8.
The square root of 49 is 7 and half of 14 is 7.
Thesquare root of 144 is 12 and half of 24 is 12.

On each "perfect square trinomial" on the right side, they are all in the form where the square root of the last number is half the middle number. So if you have a perfect square trinomial, which is always the case in completing the square, you can easily simplify it.

Here is the **Baldwin Method of Completing the Square**. We will use our quadratic for this process, it'll do, as will ANY quadratic. We have $f(x) = 2x^2 - 5x - 12$.

1] Factor out the first term's coefficient: $f(x) = 2[x^2 - \frac{5}{2}x \qquad\qquad - 6]$

Notice I left room between the 2nd and 3rd terms.

2] Now the four main steps: Using the 2nd term's coefficient, cut it in half, square it, add it, and subtract it.

$$-\frac{5}{2}\times\frac{1}{2}=-\frac{5}{4} \qquad \left(-\frac{5}{4}\right)^2=\frac{25}{16}$$ Add it in and subtract it out.

$$f(x)=2[x^2-\frac{5}{2}x+\frac{25}{16}-\frac{25}{16}-6]$$

3] Recombining gives us $f(x)=2[(x-\frac{5}{4})^2-\frac{25}{16}-\frac{96}{16}]$, and simplifying gives us $f(x)=2[(x-\frac{5}{4})^2-\frac{121}{16}]$, and the 2 can be distributed to $f(x)=2(x-\frac{5}{4})^2-\frac{121}{8}$ or $f(x)=2(x-1\frac{1}{4})^2-15\frac{1}{8}$.

This is the **AHK Form**, or **Vertex Form**, being that it is $y=a(x-h)^2+k$ where (h, k) is the vertex of the parabola. You can see that $f(x)=2(x-1\frac{1}{4})^2-15\frac{1}{8}$ gives us a vertex of $V(1\frac{1}{4},-15\frac{1}{8})$.

4] If you were going to use this method to find *x*-intercepts [not the best method for this activity, but it still works], Just set *y* = 0 and solve for *x*.

$$0=2(x-\frac{5}{4})^2-\frac{121}{8}$$
$$2(x-\frac{5}{4})^2=\frac{121}{8}$$
$$(x-\frac{5}{4})^2=\frac{121}{16}$$
$$x-\frac{5}{4}=\pm\sqrt{\frac{121}{16}}$$
$$x=\frac{5}{4}\pm\sqrt{\frac{121}{16}}$$
$$x=\frac{5}{4}\pm\frac{11}{4}=\frac{16}{4} \text{ or } \frac{-6}{4}$$
$$x=4 \text{ or } -\frac{3}{2}$$

Notice the x-intercepts are the same with each method.